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PRIME IDEALS AND RADICALS OF CENTRED EXTENSIONS AND TENSOR PRODUCTS*

BY

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ABSTRACT

In this paper, we study prime ideals and radicals of centred extensions of rings. Obtained results are applied to tensor products of algebras over commutative rings.

0. Introduction

Several authors studied prime ideals and related objects of the tensor product $A \otimes_F B$ of F-algebras A and B over a field F (cf. [1, 6, 7, 8]). In particular, J. Krempa obtained conditions under which the tensor product $A \otimes_F B$ is prime, semiprime and, more generally, S-semisimple for some radical properties S. A main tool used in his paper is the Martindale ring of quotients of A and B and the corresponding extended centroids.

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On the other hand, prime ideals and radicals of centred extensions were studied in [2, 3, 4, 11, 12]. In the last two of the quoted papers the authors considered finitely generated extensions whereas in [2, 3] the general case was studied.

In this paper we obtain some new results on prime ideals and radicals of centred extensions and apply them to tensor products of algebras over commutative rings.

In Section 1 we study prime ideals and radicals of a centred extension S of a prime ring R. It is known that there is a canonical torsion-free extension S^* of S which is an extension of Q, the Martindale ring of quotients of R, and the centralizer $V = V_{S^*}(Q)$ of Q in S^* is a C-algebra, where C is the extended centroid of R. There is a one-to-one correspondence between the set of all the R-disjoint prime ideals of S and the set of all the prime ideals of V (see [3], Sections 2 and 5). The main results in Section 1 give a relation between $\alpha(S)$ and $\alpha(V)$ for several radicals α including the prime, locally nilpotent, strongly prime and Jacobson radicals. Also, we prove that a prime ideal P of S has some special property (e.g. is strongly prime, non-singular, locally nilpotent-semisimple) if R and the corresponding prime ideal P_0 of V have the same property.

In Section 2 we apply the results obtained in Section 1 to study A-B-disjoint prime ideals of $A \otimes_F B$, where A and B are algebras over a commutative ring F. We prove, for instance, that there is a one-to-one correspondence between the A-B-disjoint prime ideals of $A \otimes_F B$ and the prime ideals of $C(A) \otimes_F C(B)$, where $C(A)$ and $C(B)$ are the extended centroids of the prime F-algebras A and B, respectively. In particular, this gives a characterization for $A \otimes_F B$ to be prime and extends some results of [8, 6]. Also, we find conditions under which an A-B-disjoint prime ideal is strongly prime, non-singular, locally nilpotent-semisimple and primitive, among others. Finally, we prove that if A and B are prime α-semisimple, then $\alpha(A \otimes B)$ equals the prime radical of $A \otimes B$, for several radicals α. The corresponding result for the Jacobson radical is also obtained.

In Section 3 we study prime (not necessarily disjoint) ideals of $A \otimes_F B$. We give an equivalent condition for $A \otimes_F B$ to be prime and we show that a prime ideal P of $A \otimes_F B$ is strongly prime (non-singular, locally nilpotent-semisimple) if and only if $P_A = \{a \in A \mid a \otimes 1 \in P\}$ and $P_B = \{b \in B \mid 1 \otimes b \in P\}$ are strongly prime (non-singular, locally nilpotent-semisimple) ideals of A and B, respectively. Also a sufficient condition for P to be primitive is obtained. Finally, a sufficient condition for $A \otimes_F B$ to be α-semisimple, for several radicals α, is given.

Throughout this paper every ring has an identity element and if $R \subseteq S$ is a ring extension, then R and S share the same identity. If I is a two-sided ideal of a ring R we write $I \lhd R$ and we simply say that I is an ideal of R. Finally, module and submodule mean bimodule and sub-bimodule, respectively.

1. Centred extensions

We first recall some results from [3] (see also [2]) that we will need in this paper.

Let $R \subseteq S$ be an extension of rings sharing the identity 1. We say that S is a **centred extension** of R if S contains a subset X such that $1 \in X$, S is generated by X as an R-module and $rx = xr$ for all $r \in R$, $x \in X$. In what follows we denote S by $R[X]$.

Let R be a prime ring, Q the Martindale ring of right quotients of R and C the extended centroid of R, i.e., the center of Q. Denote by (S^*, j) the **canonical torsion-free extension** of S defined in ([3], §2.2 and §5). Recall that S^* is an extension of Q generated as a Q-module by the centralizing set X and $j\colon S \to S^*$ is a ring and an R-bimodule homomorphism. There exists a subset E of X which contains 1 and which is a basis of S^* over Q. This subset E can be chosen as a maximal R-independent subset of X containing 1. The centralizer $V = V_{S^*}(Q)$ of Q in S^* is a C-algebra. We have $V = \sum_{e \in E} \oplus Ce$.

Given an R-submodule N of S, the **closure** $[N]$ of N is defined as the submodule

$$[N] = \{x \in S \mid \text{ there exists } 0 \neq I \lhd R \text{ with } xI \subseteq N\}.$$

It is known that

$$[N] = \{y \in S \mid \text{ there exists } 0 \neq H \lhd R \text{ with } Hy \subseteq N\}.$$

The submodule N is said to be **closed** if $[N] = N$.

A submodule M of S is said to be **dense** in S if $[M] = S$. It is known that if E is a maximal R-independent subset of X, then $M = \sum_{e \in E} Re$ is a free dense submodule of S.

A one-to-one correspondence between the following sets was established in ([3], Theorem 2.15):

(i) The set of all the closed R-submodules of S.

(ii) The set of all the closed Q-submodules of S^*.

(iii) The set of all the C-subspaces of V.

This correspondence associates the closed R-submodule N of S with the closed Q-submodule N^* of S^* and the C-subspace N_0 of V if $j^{-1}(N^*) = N$ and $N^* = QN_0$. If j is an embedding (e.g. if S is torsion-free as a right R-module which, for instance, holds when S is prime), then $N = QN_0 \cap S$. Moreover, the correspondence preserves right closed ideals, two-sided closed ideals and R-disjoint prime ideals ([3], §5). Finally, using ([3], Corollary 2.16) is easy to see that it also preserves intersections and inclusions.

Throughout this section we use the notation and terminology introduced in [3]. In particular, a submodule N of S is said to be R-disjoint if $N \cap R = 0$.

For an R-disjoint prime ideal P of S we denote by P^* the extension of P to S^* and by P_0 the contraction $P^* \cap V$. As we said above, $P = j^{-1}(P^*)$ and $P^* = QP_0$.

Remark 1.1: To study the factor rings S/P, S^*/P^* and V/P_0 we may factor out the ideals P, P^* and P_0, respectively, and assume that they are equal to zero.

Indeed, S/P is a centred extension of R with the generating set $\{x+P \mid x \in X\}$ and the canonical torsion-free extension $(S/P)^*$ is isomorphic to S^*/P^* as a ring and a Q-bimodule ([3], Lemma 3.4). The map j: $S \to S^*$ induces a homomorphism j': $S/P \to S^*/P^*$, which is injective since $j^{-1}(P^*) = P$. Moreover S^*/P^* is a free Q-module with the basis $E' = \{e+P^* \mid e \in X'\}$, for a subset $X' \subseteq X$. We shall show that $V_{(S^*/P^*)}(Q) \simeq V/P_0$. Obviously $V/P_0 \simeq (V+P^*)/P^* \subseteq S^*/P^*$ and for $v \in V$, $q \in Q$, $q(v+P^*) = (v+P^*)q$. Hence $(V+P^*)/P^* \subseteq V_{S^*/P^*}(Q)$. Now if $\bar{s} \in V_{S^*/P^*}(Q)$, then $\bar{s} = \sum q_i(e_i+P^*)$, for some $q_i \in Q$, $e_i + P^* \in E'$ and $q\bar{s} = \bar{s}q$ for all $q \in Q$. Clearly $q_i \in C$ and $e_i + P^* \in (V+P^*)/P^*$. Consequently $\bar{s} \in \sum C(e_i+P^*) \subseteq (V+P^*)/P^*$. Thus $V/P_0 \simeq (V+P^*)/P^* = V_{S^*/P^*}(Q)$ and the remark follows.

Now we obtain some results on prime ideals and radicals. We start with the prime radical β.

PROPOSITION 1.2: *If R is a prime ring and $S = R[X]$ is a torsion-free centred extension of R, then $\beta(S) = Q\beta(V) \cap S$.*

Proof: The ring S is torsion-free, so for every prime ideal P_0 of V, $QP_0 \cap S$ is an R-disjoint prime ideal of S. Hence, since the above described correspondence preserves inclusion, $\beta(S) \subseteq \bigcap\{QP_0 \cap S \mid P_0 \text{ is a prime ideal of } V\} = Q\beta(V) \cap S$.

Applying Zorn's Lemma, we can find an ideal M of V maximal among ideals $I \subseteq \beta(V)$ such that $QI \cap S \subseteq \beta(S)$. If $M \neq \beta(V)$, then V/M contains a non-zero nilpotent ideal N/M. Suppose $N^n \subseteq M$. Since elements of Q and V commute, $(QN)^n \subseteq QN^n \subseteq QM$. Hence $(QN \cap S)^n \subseteq QM \cap S \subseteq \beta(S)$ and therefore $QN \cap S \subseteq \beta(S)$, a contradiction. ∎

Proposition 1.2 immediately implies the following corollary which was proved in ([3], §7) only under some finiteness assumption.

COROLLARY 1.3: *Suppose that R is a prime ring and $S = R[X]$ is a centred extension of R with* $\operatorname{Ker} j \subseteq \beta(S)$. *Then $\beta(S) = j^{-1}(Q\beta(V))$, where $j\colon S \to S^*$ is the canonical mapping. In particular, in this case $\beta(S)$ is the intersection of R-disjoint prime ideals of S.*

Proof: Apply Proposition 1.2 to the torsion-free centred extension $S/\operatorname{Ker} j$. ∎

The following example shows we cannot expect that the prime radical be always an intersection of R-disjoint prime ideals.

Example 1.4: Let $S = \mathbf{Z}[X]/ < 2X >$, where $\mathbf{Z}$ is the ring of integer numbers. Then $\beta(S) = 0$ and the intersection of $\mathbf{Z}$-disjoint prime ideals of S is the ideal generated by X. In this example $S^* = V = \mathbf{Q}$ and $j^{-1}(0) = XS$, where $\mathbf{Q}$ is the field of rational numbers.

Since $V \subseteq S^*$ and S^* is free over Q, the following is clear.

LEMMA 1.5: *Suppose that $v \in V$ and $q \in Q$. Then $vq = 0$ if and only if either $v = 0$ or $q = 0$.*

The **locally nilpotent** (resp. **nil**) radical $L(A)$ (resp. $\operatorname{Nil}(A)$) of a ring A is the largest locally nilpotent (resp. nil) ideal of A. We say that a prime ideal P of a ring A is **locally nilpotent** (resp. **nil**) **semisimple** (**l.n-semisimple**, for short) if $L(A/P) = 0$ (resp. $\operatorname{Nil}(A/P) = 0$).

PROPOSITION 1.6: *Assume that R is a prime l.n.-semisimple ring and $S = R[X]$ is a torsion-free centred extension of R. Then $L(S) = QL(V) \cap S$.*

Proof: Take $y_1, \ldots, y_k \in L(V)$ and $q_1, \ldots, q_k \in Q$. There exists an integer t such that each product of t elements from the set $\{y_1, \ldots, y_k\}$ is equal to zero. Elements from V commute with those from Q. Hence each product of t elements from the set $\{y_1q_1, \ldots, y_kq_k\}$ is equal to zero. This implies that $QL(V)$ is a locally nilpotent ideal of S^*. Consequently $QL(V) \cap S \subseteq L(S)$.

We have $[L(S)] = QI_0 \cap S$ for an ideal I_0 of V. Take $y_1, \ldots, y_n \in I_0$. Applying ([3], Lemma 1.1 (ii) and Corollary 2.16) one gets that there exists $0 \neq H \lhd R$ such that $y_j H \subseteq L(S)$, $1 \leq j \leq n$. Since $L(R) = 0$, there are $h_1, \ldots, h_m \in H$ such that for every integer p the product of some p elements from the set $\{h_1, \ldots, h_m\}$ is non-zero. However $T = \{y_j h_i \mid 1 \leq j \leq n, 1 \leq i \leq m\} \subseteq L(S)$, so there exists an integer t such that each product of t elements from T is equal to zero. Suppose that $h_{i_1} \cdots h_{i_t} \neq 0$, $i_1, \ldots, i_t \in \{1, \ldots, m\}$. For any $j_1, \ldots, j_t \in \{1, \ldots, n\}$, $(y_{j_1} \cdots y_{j_t})(h_{i_1} \cdots h_{i_t}) = y_{j_1} h_{i_1} \cdots y_{j_t} h_{i_t} = 0$. Hence, applying Lemma 1.5, we obtain that $y_{j_1} \cdots y_{j_t} = 0$. This proves that I_0 is locally nilpotent. Consequently $L(S) \subseteq QI_0 \cap S \subseteq QL(V) \cap S$ and we are done. ∎

The following corollary is clear.

COROLLARY 1.7: *Suppose that R is a prime l.n.-semisimple ring and $S = R[X]$ is a centred extension of R with* $\operatorname{Ker} j \subseteq L(S)$. *Then $L(S) = j^{-1}(QL(V))$. In particular, in this case $L(S)$ is the intersection of R-disjoint l.n.-semisimple prime ideals.*

COROLLARY 1.8: *Assume that R is a prime ring and $S = R[X]$ is a centred extension of R. An R-disjoint prime ideal P of S is l.n.-semisimple if and only if R is l.n.-semisimple and P_0 is an l.n.-semisimple ideal of V.*

Proof: Applying Remark 1.1 we may assume that $P = P_0 = 0$.

Suppose that $L(S) = 0$. Since elements from X commute with those from R, $L(R)[X]$ is a locally nilpotent ideal of S. Hence $L(R) = 0$ and Proposition 1.6 implies that $L(V) = 0$. The converse implication is a direct consequence of Proposition 1.6. ∎

Concerning the nil radical we have the following

PROPOSITION 1.9: *Assume that R is a prime nil-semisimple ring and $S = R[X]$ is a torsion-free centred extension of R. Then* $\operatorname{Nil}(S) \subseteq Q\operatorname{Nil}(V) \cap S$. *If, in addition, X is a commuting set, then* $\operatorname{Nil}(S) = Q\operatorname{Nil}(V) \cap S = \beta(S)$.

Proof: We have $[\operatorname{Nil}(S)] = QN_0 \cap S$ for an ideal N_0 of V. By Corollary 2.16 of [3], for every $x \in N_0$ there exists $0 \neq H \lhd R$ with $xH \subseteq \operatorname{Nil}(S)$. Since $\operatorname{Nil}(R) = 0$, there exists $h \in H$ such that $h^n \neq 0$ for every natural number n. However $xh \in \operatorname{Nil}(S)$ and x and h commute, so $0 = (xh)^m = x^m h^m$ for a natural number m. Applying Lemma 1.5 we get that $x^m = 0$. This shows that N_0 is a nil ideal of V. Consequently $\operatorname{Nil}(S) \subseteq Q\operatorname{Nil}(V) \cap S$.

If X is a commuting set, then the ring V is commutative. Hence $\mathrm{Nil}(V) = \beta(V)$ and $\beta(S) \subseteq \mathrm{Nil}(S) \subseteq Q\,\mathrm{Nil}(V) \cap S = Q\beta(V) \cap S = \beta(S)$, by Proposition 1.2. The proof is complete. ∎

The following corollaries are clear.

COROLLARY 1.10: *If R is a prime nil-semisimple ring and $S = R[X]$ is a centred extension of R, then $\mathrm{Nil}(S) \subseteq j^{-1}(Q\,\mathrm{Nil}(V))$. If, in addition, the set X is commuting and $\mathrm{Ker}\, j \subseteq \mathrm{Nil}(S)$, then $\mathrm{Nil}(S) = j^{-1}(Q\,\mathrm{Nil}(V))$.*

COROLLARY 1.11: *Let R be a prime ring and $S = R[X]$ a centred extension of R. Assume that R is nil-semisimple and P is an R-disjoint prime ideal of S such that P_0 is nil-semisimple. Then P is nil-semisimple. The converse holds provided X is a commuting set.*

In Section 2 we give an example (Example 2.15) which in particular shows that there exists a centred extension S of a prime ring R with $\mathrm{Nil}(R) \neq 0$ and a prime nil-semisimple ideal P of S such that P_0 is not nil-semisimple.

Recall that a ring A is said to be (right) **strongly prime** if every non-zero ideal I of A contains a finite set F, called an **insulator** of I, such that $r_A(F) = \{a \in A \mid Fa = 0\} = 0$. An ideal P of A is called (right) **strongly prime** if the ring A/P is strongly prime ([9]).

PROPOSITION 1.12: *Suppose that R is a prime ring and $S = R[X]$ is a centred extension of R. Then an R-disjoint prime ideal P of S is strongly prime if and only if R is strongly prime and P_0 is a strongly prime ideal of V.*

Proof: By Remark 1.1 we may assume that $P = P^* = P_0 = 0$.

Suppose that S is strongly prime. Then R is strongly prime by ([3], Proposition 6.5). Take $0 \neq I_0 \lhd V$. Then $0 \neq I = QI_0 \cap S \lhd S$. Hence there exists $F = \{y_1, \ldots, y_n\} \subseteq I$ such that $r_S(F) = 0$. For every $1 \leq i \leq n$ there are $q_{ij} \in Q$, $m_{ij} \in I_0$, $1 \leq j \leq t_i$, such that $y_i = \sum_j q_{ij} m_{ij}$. Let $v \in r_V(\{m_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq t_i\})$. Obviously $Fv = 0$. Take $0 \neq H \lhd R$ with $vH \subseteq S$. Then $FvH = 0$ and so $vH = 0$. This implies that $v = 0$. Consequently $\{m_{ij}\}$ is an insulator of I_0. This shows that V is strongly prime.

Conversely, suppose that R and V are strongly prime and take $0 \neq I \lhd S$. If $I \cap R \neq 0$, then there exists a finite set $F \subseteq I$ such that $r_R(F) = 0$. Take $s \in S$ with $Fs = 0$. Since $S \subseteq S^*$ and S^* is free over Q, we may write $s = q_1 e_1 + \cdots + q_n e_n$, where $\{e_1, \ldots, e_n\}$ are free over Q and $q_i \in Q$, $1 \leq i \leq n$. Also there exists $0 \neq H \lhd R$ such that $q_iH \subseteq R$, for $1 \leq i \leq n$. Thus $Fq_iH = 0$, so $q_iH = 0$ and it follows that $q_i = 0$. Consequently $s = 0$ and hence $r_S(F) = 0$.

Hence we may assume that $I \cap R = 0$. There exists $0 \neq I_0 \lhd V$ with $[I] = QI_0 \cap S$. Let F be an insulator of I_0. By ([3], Lemma 1.1 (ii) and Corollary 2.16) there exists $0 \neq H \lhd R$ with $FH \subseteq I$. Since R is strongly prime, we can find a finite set $F' \subseteq H$ such that $r_R(F') = 0$.

We shall show that $FF' = F'F \subseteq I$ is an insulator of I. Note that if $s \in S$ and $F'Fs = 0$, then since $Fs \subseteq S^*$ and $r_R(F') = 0$, we obtain as above that $Fs = 0$. Consequently $A = r_S(FF') = r_S(F)$. Moreover A is a left R-submodule and a right ideal of S. Let A_0 be a subspace of V such that $[A] = QA_0 \cap S$. For every $y \in A_0$ there exists $0 \neq J \lhd R$ with $yJ \subseteq A$. Now $FyJ = 0$ which implies that $Fy = 0$. Thus $y \in r_V(F) = 0$. Consequently $A_0 = 0$ and it follows that $A = 0$. The proof is complete. ∎

Recall that the **strongly prime radical** of a ring A is defined as $s(A) = \bigcap\{I \lhd A \mid A/I \text{ is a strongly prime ring}\}$. In the next Corollary we shall use the characterization of $s(A)$ as the largest ideal of A which does not contain an insulator modulo any ideal of A ([5], Lemma 1.1).

COROLLARY 1.13: *If R is a strongly prime ring and $S = R[X]$ is a torsion-free centred extension of R, then $s(S) = Qs(V) \cap S$.*

Proof: From Proposition 1.12 it follows that $s(S) \subseteq Qs(V) \cap S$. If $s(S) \neq Qs(V) \cap S$, then there exist an ideal I of S and a finite subset F of $Qs(V) \cap S$ such that $Fs \subseteq I$, $s \in S$, imply $s \in I$. Assume that $F = \{y_1, \ldots, y_n\}$, where $y_i = \sum q_{ij}v_{ij}$ for some $q_{ij} \in Q$, $v_{ij} \in s(V)$. Let $[I] = QI_0 \cap S$ for an ideal I_0 of V. In view of the quoted result of [5], to get a contradiction, it suffices to show that $\{v_{ij} \mid \text{all } i,j\}$ is an insulator of $s(V)$ modulo I_0. Take $v \in V$ such that $v_{ij}v \in I_0$ for all i,j. By ([3], Lemma 1.1 (ii) and Corollary 2.16) there exists $0 \neq H \lhd R$ such that $v_{ij}vH \subseteq I$ for all i,j and $vH \subseteq S$. There also exists $0 \neq H' \lhd R$ such that for all i,j, $q_{ij}H' \subseteq S$. Since R is a prime ring, $T = H'H$ is a non-zero ideal of R. Moreover, for all i,j, $q_{ij}v_{ij}vT = q_{ij}v_{ij}vH'H = q_{ij}H'v_{ij}vH \subseteq SI \subseteq I$ which implies that $FvT \subseteq I$. Hence, since $vT \subseteq S$ and F is an insulator of S modulo I, we get $vT \subseteq I$. Consequently $v \in [I]^* \cap V = QI_0 \cap V = I_0$ and the proof is complete. ∎

COROLLARY 1.14: *Suppose that R is a strongly prime ring and $S = R[X]$ is a centred extension of R with* $\operatorname{Ker} j \subseteq s(S)$. *Then $s(S) = j^{-1}(Qs(V))$. In* *particular, in this case $s(S)$ is the intersection of R-disjoint strongly prime ideals of S.*

Now we consider non-singular prime ideals. Recall that the (right) **singular ideal** of a ring A is defined by $Z(A) = \{a \in A \mid r_A(a)$ is an essential right ideal of $A\}$. The ring A (resp. an ideal I of A) is said to be (right) **non-singular** if $Z(A) = 0$ (resp. $Z(A/I) = 0$).

PROPOSITION 1.15: *Suppose that R is a non-singular prime ring and $S = R[X]$ is a centred extension of R. If P is an R-disjoint prime ideal of S such that P_0 is a non-singular ideal of V, then P is a non-singular ideal of S.*

Proof: Applying Remark 1.1 we may assume $P = P_0 = 0$, so that V is a non-singular ring.

Suppose that $Z(S) \neq 0$. By ([3], Corollary 1.8 (iii) there exists $0 \neq s \in Z(S) \cap M$, where $M = \sum_{e \in E} Re$ is a free dense R-submodule of S. We may assume that $s = a_1e_1 + \cdots + a_ne_n$ is of minimal support. Then by ([3], Lemma 2.1) $s = a_1m$, for some $0 \neq m \in V$. Since $Z(V) = 0$, there exists a non-zero right ideal I_0 of V such that $r_V(m) \cap I_0 = 0$. Let $I = QI_0 \cap S$ and $I' = r_I(m) = \{z \in I \mid mz = 0\}$. Observe that I' is a closed left R-submodule and right ideal of S. Thus $I' = QI_0' \cap S$, where I_0' is a right ideal of V contained in I_0. For every $y \in I_0'$ there exists $0 \neq H \lhd R$ with $yH \subseteq I'$. Then $myH = 0$ and so $my = 0$. Since $r_V(m) \cap I_0 = 0$, it follows that $y = 0$. Consequently $r_I(m) = 0$.

Now, $a_1 \notin Z(R)$, so there exists a non-zero right ideal H of R such that $r_R(a_1) \cap H = 0$. The ring S is prime and $0 \neq I \lhd S$, so HI is a non-zero right ideal of S. Hence, since $s \in Z(S)$, there exists $0 \neq z \in HI$ with $sz = 0$. Let $z = \sum h_tx_t$, where $h_t \in H$ and $x_t \in X$. Then $mz = m\sum h_tx_t = \sum h_tmx_t$. Since E is a Q-basis of S^*, for each t, $mx_t = \sum_l q_{tl}e_l$, where $q_{tl} \in Q$ and $e_l \in E$. There exists $0 \neq A \lhd R$ such that $q_{tl}A \subseteq R$ for all t, l. Clearly $mzA \subseteq \sum_{e \in E} He$. Moreover $a_1mzA = szA = 0$. Hence, since the set E is R-independent and $r_R(a_1) \cap H = 0$ we obtain $mzA = 0$. However $zA \subseteq HIA \subseteq I$ and $r_I(m) = 0$, so $zA = 0$. Therefore $z = 0$, a contradiction. ∎

Remark 1.16: We do not know whether the converse of the above proposition holds. If P is a non-singular R-disjoint prime ideal of S, then R is non-singular ([3], Proposition 6.6). However we do not know whether P_0 has to be necessarily non-singular in this case.

Notice that the class of prime rings A with $Z(A) = 0$ is a special class of rings. Hence $z(A) = \bigcap\{P \mid P$ is a prime ideal of A such that $Z(A/P) = 0\}$ defines a radical of A. We have

LEMMA 1.17: *If A is a strongly prime ring, then $Z(A) = 0$.*

Proof: If $Z(A) \neq 0$, then there exists a finite set $F \subseteq Z(A)$ such that $r_A(F) = 0$. However $r_A(F) = \bigcap\{r_A(a) \mid a \in F\}$ is an essential right ideal of A, a contradiction. ∎

Lemma 1.17 immediately gives

COROLLARY 1.18: *For every ring A, $z(A) \subseteq s(A)$.*

Applying Proposition 1.15 we obtain

COROLLARY 1.19: *Suppose that R is a prime non-singular ring and $S = R[X]$ is a centred extension of R. Then $z(S) \subseteq j^{-1}(Qz(V))$. In particular, if S is torsion-free, then $z(S) \subseteq Qz(V) \cap S$.*

Recall that for a ring A with an identity the **Brown–McCoy radical** $G(A)$ of A is defined as the intersection of all maximal ideals of A.

Let R be a simple ring and $S = R[X]$ a torsion-free centred extension of R. Obviously each prime ideal of S is R-disjoint. The correspondence between the R-disjoint prime ideals of S and prime ideals of V immediately implies that P is a maximal ideal of S if and only if P_0 is a maximal ideal of V. This gives

COROLLARY 1.20: *If $S = R[X]$ is a centred extension of a simple ring R, then $G(S) = j^{-1}(QG(V))$. In particular, if S is torsion-free, then $G(S) = QG(V) \cap S$.*

A centred extension S of a ring R is called **free** if we can find a set of centralizing generators X which is R-independent [2].

If $S = R[X]$ is a free centred extension of a primitive ring R and P is an R-disjoint prime ideal of S, then P is a primitive ideal of S provided P_0 is a primitive ideal of V ([4], Corollary 4). This result is also true for not necessarily free centred extensions.

PROPOSITION 1.21: *Assume that R is a (right) primitive ring and $S = R[X]$ is a centred extension of R. If P is an R-disjoint prime ideal of S such that P_0 is a primitive ideal of V, then P is a primitive ideal of S.*

Proof: Let T be the polynomial ring over R in non-commuting indeterminates $\{Y_x \mid x \in X\}$ and let $f: T \to S$ be the R-epimorphism defined by $f(Y_x) = x$. Certainly T is a free centred extension of R and $f^{-1}(P)$ is an R-disjoint prime ideal of T. Moreover $S/P \simeq T/f^{-1}(P)$ and so $(S/P)^* \simeq (T/f^{-1}(P))^* \simeq T^*/(f^{-1}(P))^*$. Therefore $V_{(T/f^{-1}(P))^*}(Q) \simeq V_{(S/P)^*}(Q) \simeq V/P_0$. Since P_0 is a primitive ideal of V, $V_{(T/f^{-1}(P))^*}(Q)$ is a primitive ring. By Remark 1.1, $V_{(T/f^{-1}(P))^*}(Q) \simeq V_{T^*}(Q)/(f^{-1}(P))_0$, so $(f^{-1}(P))_0$ is a primitive ideal of $V_{T^*}(Q)$. Since T is a free centred extension of R, Corollary 4 of [4] implies that $f^{-1}(P)$ is a primitive ideal of T. Consequently P is a primitive ideal of S. ∎

From Proposition 1.21 we immediately get

COROLLARY 1.22: *Assume that R is a primitive ring and $S = R[X]$ is a centred extension of R. Then $J(S) \subseteq j^{-1}(QJ(V))$, where J denotes the Jacobson radical. In particular, if S is torsion-free, then $J(S) \subseteq QJ(V) \cap S$.*

The main result in [4] shows that if R is a Jacobson semisimple ring and $S = R[X]$ is a free centred extension of R such that $C[X]$ is Jacobson semisimple for every field C which is the extended centroid of a primitive factor of R, then S is also Jacobson semisimple. Using the results of this section one can easily see that the same holds for several other radicals. To get this it suffices to note that if $R[X]$ is a free centred extension of R, then for each ideal I of R, $I[X]$ is an ideal of $R[X]$ and $R[X]/I[X] \simeq (R/I)[X]$ in a natural way, and $(R/I)[X]$ is a free centred extension of R/I.

COROLLARY 1.23: *Let $S = R[X]$ be a free centred extension of a ring R. If $\alpha(R) = 0$ and $\alpha(C[X]) = 0$ for every field C which is the extended centroid of a prime factor of R, then $\alpha(S) = 0$, where $\alpha = s, z, \beta, L, \mathrm{Nil}, J, G$.*

2. Tensor products: disjoint case

Throughout this section we assume that A and B are algebras over a commutative ring D. For an ideal I of $A \otimes_D B$ we put

$$I_A = \{a \in A \mid a \otimes 1 \in I\}$$

and

$$I_B = \{b \in B \mid 1 \otimes b \in I\}.$$

The ideal I is said to be A-B-disjoint if $I_A = I_B = 0$.

Note that if P is a prime ideal of $A \otimes_D B$, then P_A and P_B are prime ideals of A and B, respectively. In this section we study A-B-disjoint prime ideals of $A \otimes_D B$.

Assume that P is an A-B-disjoint prime ideal of $A \otimes_D B$. Then A and B are prime rings and the canonical mappings $a \to a \otimes 1$ and $b \to 1 \otimes b$ are embeddings of A and B into $A \otimes_D B$, respectively. Note that $I_D = \{d \in D \mid d \otimes 1 = 0\} = \{d \in D \mid 1 \otimes d = 0\}$ is a prime ideal of D and there exists a canonical ring isomorphism $A \otimes_D B \simeq A \otimes_{D/I_D} B$. Moreover D/I_D is a commutative domain isomorphic to a subring of the center of A and a subring of the center of B. Thus studying A-B-disjoint prime ideals of $A \otimes_D B$ we can assume without loss of generality that A and B are prime D-algebras and D is a commutative domain contained in the center of A and in the center of B. If there is no danger of misunderstanding, instead of $A \otimes_D B$ we write $A \otimes B$ or denote it by S. Now we establish some elementary properties of S.

Lemma 2.1: *Suppose A and B are prime D-algebras where D is a commutative domain contained in the centers of A and B. If $0 \neq a \in A$ and $0 \neq b \in B$, then $a \otimes b \neq 0$ in S.*

Proof: From the universal property of tensor products it follows that $A \otimes F \simeq D^{-1}A$, where F is the field of fractions of D and $D^{-1}A$ denotes the localization of A at $D - \{0\}$. Similarly, $F \otimes B \simeq D^{-1}B$. In particular we have $a \otimes 1 \neq 0$ in $A \otimes F$ and $1 \otimes b \neq 0$ in $F \otimes B$. However F is a field, so $(a \otimes 1) \otimes (1 \otimes b) \neq 0$ in $(A \otimes F) \otimes_F (F \otimes B) \simeq A \otimes B \otimes F$. This implies that $a \otimes b \otimes 1 \neq 0$ in $A \otimes B \otimes F$ and consequently $a \otimes b \neq 0$ in $A \otimes B$. ∎

Lemma 2.2: *Suppose A and B are prime D-algebras where D is a commutative domain contained in the centers of A and B. If P is an ideal of S which is maximal among the ideals of S not containing non-zero elements of the type $a \otimes b$, $a \in A$, $b \in B$, then P is an A-B-disjoint prime ideal of S.*

Proof: Suppose that I and J are ideals of S strictly containing P. By maximality of P, there are $0 \neq a \otimes b \in I$ and $0 \neq c \otimes d \in J$. Since A and B are prime, $axc \neq 0$ and $byd \neq 0$ for some $x \in A$, $y \in B$. Now by Lemma 2.1, $0 \neq axc \otimes byd \in IJ$. Hence $IJ \not\subseteq P$ and P is prime. If $a \otimes 1 \in P$, $a \in A$, we have that $a \otimes 1 = 0$ and so using Lemma 2.1 we obtain $a = 0$. Thus $P_A = 0$. Similarly $P_B = 0$ and the proof is complete. ∎

The above shows that there exists an A-B-disjoint prime ideal of S if and only if A and B are prime D-algebras and D is a commutative domain contained in the centers of A and B. Moreover, in this case we may identify A with $A \otimes 1$ and B with $1 \otimes B$ and assume that A and B are contained in S. We keep these assumptions throughout the rest of this section.

LEMMA 2.3: *If P is an A-B-disjoint prime ideal of S and $a \otimes b \in P$ for some $a \in A$, $b \in B$, then either $a = 0$ or $b = 0$.*

Proof: Note that $(a \otimes 1)(A \otimes B)(1 \otimes b) = B(a \otimes 1)(1 \otimes b)A = B(a \otimes b)A$. Hence, if $a \otimes b \in P$, then either $a \otimes 1 \in P$ or $1 \otimes b \in P$, i.e., either $a \in P_A = 0$ or $b \in P_B = 0$. ∎

The proof of the following lemma is straightforward.

LEMMA 2.4: *Let F be the field of fractions of D. If E is a maximal D-independent subset of B, then $\{1 \otimes e\}_{e \in E}$ is an F-basis of $F \otimes B$.*

Let $Q = Q(A)$ be the Martindale ring of right quotients of A and $C = C(A)$ the extended centroid of A, i.e., the center of Q. Since C is a field and $D \subseteq C$, the field of fractions F of D is contained in C. The canonical isomorphism $C \otimes B \simeq C \otimes_F F \otimes_D B$ and Lemma 2.4 imply that $\{1 \otimes e\}_{e \in E}$ is a C-basis of $C \otimes B$, where E is a maximal D-independent subset of B. Hence $Q \otimes B \simeq Q \otimes_C C \otimes_D B$ is also a free Q-module with the basis $\{1 \otimes e\}_{e \in E}$. This also shows that the canonical map of $C \otimes B$ into $Q \otimes B$ is an embedding. We can obviously assume that $1 \in E$ and hence that $Q \subseteq Q \otimes B$ in a canonical way. Hereafter we denote by E the above constructed Q-basis $\{1 \otimes e\}_{e \in E}$ of $Q \otimes B$.

Now we are ready to apply the results on centred extensions to study A-B-disjoint prime ideals of S.

Note that S is a centred extension $A[X]$ of A, where A is identified with $A \otimes 1$ and $X = \{1 \otimes b \mid b \in B\}$. In what follows we put

$$T = [0] = \{x \in S \mid xH = 0 \text{ for some } 0 \neq H \triangleleft A\}$$

and call T the A-**torsion ideal** of S. Clearly every A-disjoint prime ideal of S contains T.

The canonical map of $A \otimes B$ into $Q \otimes B$ will be denoted by h. Let us point out that since $Q \otimes B$ is free as a Q-module, $Q \otimes B$ is A-torsion-free in the sense of ([3], Section 2).

Let (S^*, j) be the canonical torsion-free extension of S. We have

THEOREM 2.5: *Under the above notation there exists an isomorphism of Q-extensions ϕ: $(S^*, j) \to (Q \otimes B, h)$. Moreover,* $\operatorname{Ker} h = T$ *and* $\phi(V) = C \otimes B$, *where* $V = V_{S^*}(Q)$.

Proof: By the universal property of (S^*, j) there exists an A-homomorphism of rings ϕ: $S^* \to Q \otimes B$ such that $\phi \circ j = h$. The universal property of tensor products easily implies that there exists a Q-homomorphism of rings ψ: $Q \otimes B \to S^*$ such that $\psi(q \otimes b) = j(1 \otimes b)q$, for all $q \in Q$ and $b \in B$. Obviously, since j is an A-homomorphism, $\psi \circ h = j$. Consequently $\phi \circ \psi \circ h = h$ and $\psi \circ \phi \circ j = j$. Now using the universal properties defining tensor products and (S^*, j) we get $\phi \circ \psi = id_{Q \otimes B}$ and $\psi \circ \phi = id_{S^*}$. Consequently ϕ is an isomorphism of Q-extensions.

By ([3], Remark 2.9), $\operatorname{Ker} j = T$ and so $\operatorname{Ker} h = T$. Finally $V = V_{S^*}(Q) \simeq V_{Q \otimes B}(Q)$. However, since $C \otimes B$ and $Q \otimes B$ are C and Q-free, respectively, with the same basis E, we have $V_{Q \otimes B}(Q) = C \otimes B$ and we are done. ∎

Now as a consequence of ([3], Theorem 2.15) we obtain that there exists a one-to-one correspondence between the closed A-submodules of $A \otimes B$, the closed Q-submodules of $Q \otimes B$ and the C-subspaces of $C \otimes B$. In particular, we have (see also [3], Theorem 5.3)

COROLLARY 2.6: *There is a one-to-one correspondence between the following:*

(i) *The set of all the A-disjoint prime ideals of $A \otimes B$.*

(ii) *The set of all the Q-disjoint prime ideals of $Q \otimes B$.*

(iii) *The set of all the prime ideals of $C \otimes B$.*

This correspondence associates a prime ideal P of $A \otimes B$ with a prime ideal P^* of $Q \otimes B$ and a prime ideal P_0 of $C \otimes B$ if $h^{-1}(P^*) = P$ and $P^* = P_0(Q \otimes B)$.

Now we obtain some information about A and B-closed submodules of S.

LEMMA 2.7: *Let N be an A-closed submodule of S. Denoting by N^* the Q-closed submodule of $Q \otimes B$ corresponding to N we have*

(i) *N is a B-submodule of S if and only if N^* is a B-submodule of $Q \otimes B$.*

(ii) *N is B-disjoint if and only if N^* is B-disjoint.*

(iii) *N is B-closed if and only if N^* is B-closed.*

Proof: (i) Since $N = h^{-1}(N^*)$ and h is a B-homomorphism, it is clear that N is a B-submodule whenever N^* is a B-submodule. Assume now that N is a B-submodule and take $x = \sum_{1 \leq i \leq n} q_i \otimes b_i \in N^*$, where $q_i \in Q$, $b_i \in B$, $1 \leq i \leq n$.

There exists $0 \neq H \lhd A$ such that $q_iH \subseteq A$ for $i = 1, \ldots, n$. For $a \in H$ put $x_a = \sum_{1 \leq i \leq n} q_i a \otimes b_i \in S$. Since $h(x_a) = xa \in N^*$, we have that $x_a \in N$. Now for every $b \in B$, $x_a b \in N$, so $xba = (\sum_{1 \leq i \leq n} q_i \otimes b_i b)a = \sum_{1 \leq i \leq n} q_i a \otimes b_i b = h(x_a b) \in N^*$. Hence $xbH \subseteq N^*$ and by ([3], Corollary 2.16), $xb \in N^*$. Consequently N^* is a B-submodule of $Q \otimes B$.

(ii) For every $b \in B$ we have $1 \otimes b = h(1 \otimes b)$. Thus $1 \otimes b \in N$ in S if and only if $1 \otimes b \in N^*$ in $Q \otimes B$. This proves (ii).

(iii) If $y \in S$ and $yF \subseteq N$ for some $0 \neq F \lhd B$, then $h(y)F \subseteq N^*$. Hence if N^* is B-closed, then $h(y) \in N^*$ and $y \in N$ follows. Thus N is B-closed.

Conversely, assume that N is B-closed and take $x = \sum_{1 \leq i \leq n} q_i \otimes b_i \in Q \otimes B$ and a non-zero ideal H of A with $q_iH \subseteq A$. As in (i) we take $a \in H$ and put $x_a = \sum_{1 \leq i \leq n} q_i a \otimes b_i \in S$. Suppose there exists $0 \neq F \lhd B$ such that $xF \subseteq N^*$ and take any $b \in F$. We have $h(x_a b) = \sum_{1 \leq i \leq n} q_i a \otimes b_i b = (\sum_{1 \leq i \leq n} q_i \otimes b_i b)a = xba \in N^*$. Therefore $x_a F \subseteq N$ and so $x_a \in N$. Consequently $xa = h(x_a) \in N^*$. This shows that $xH \subseteq N^*$ and, since N^* is Q-closed, $x \in N^*$ follows. The proof is complete. ∎

Let l be the canonical homomorphism of $Q(A) \otimes B$ to $Q(A) \otimes Q(B)$. As above we have $\operatorname{Ker} l = \{y \in Q(A) \otimes B \mid yL = 0 \text{ for some } 0 \neq L \lhd B\}$. Now we shall prove

PROPOSITION 2.8: *The smallest A and B-closed submodule K of S is equal to* $\operatorname{Ker}(l \circ h)$. *In particular, K is an ideal of S.*

Proof: $Q(A) \otimes Q(B)$ is a free $Q(A)$-module as well as $Q(B)$-module. Thus the zero ideal of $Q(A) \otimes Q(B)$ is $Q(A)$ and $Q(B)$-closed. Applying twice Lemma 2.7 we obtain that $\operatorname{Ker}(l \circ h)$ is A and B-closed. Now let I be an A and B-closed submodule of S and denote by I^* the corresponding $Q(A)$-closed submodule of $Q(A) \otimes B$. By Lemma 2.7 (iii), I^* is B-closed. Also $\operatorname{Ker} l$ is the smallest B-closed submodule of $Q(A) \otimes B$. Therefore $\operatorname{Ker} l \subseteq I^*$ and $\operatorname{Ker}(l \circ h) = h^{-1}(\operatorname{Ker} l) \subseteq h^{-1}(I^*) = I$. ∎

Hereafter we denote by K the ideal $\operatorname{Ker}(l \circ h)$ and we call it the **torsion ideal** of S.

Proposition 2.8 gives

COROLLARY 2.9: *If P is an A-B-disjoint prime ideal of $A \otimes B$, then $K \subseteq P$.*

Let us notice that the canonical mapping of B into $Q(A)\otimes B$ (as well as that of B into $C(A)\otimes B$) is an embedding. Indeed, if $b\in B$ and $1\otimes b=0$ in $Q(A)\otimes B$, then $1\otimes b$, treated as an element of $A\otimes B$, belongs to T. Hence there exists $0\neq H\lhd A$ with $(1\otimes b)H=0$. Applying Lemma 2.1 we obtain $b=0$.

Now we can repeat earlier arguments to study B-disjoint prime ideals of $Q(A)\otimes B$ (resp. $C(A)\otimes B$) passing to $Q(A)\otimes Q(B)$ and $Q(A)\otimes C(B)$ (resp. $C(A)\otimes Q(B)$ and $C(A)\otimes C(B)$). Putting these together with Corollary 2.6 and Lemma 2.7 we get the following:

COROLLARY 2.10: *There is a one-to-one correspondence between the following*

(i) *The set of all the A-B-disjoint prime ideals of $A\otimes B$.*

(ii) *The set of all the $Q(A)$-$Q(B)$-disjoint prime ideals of $Q(A)\otimes Q(B)$.*

(iii) *The set of all the prime ideals of $C(A)\otimes C(B)$.*

The correspondence associates a prime ideal P of $A\otimes B$ with a prime ideal P^* of $Q(A)\otimes Q(B)$ and a prime ideal P_0 of $C(A)\otimes C(B)$ if $(l\circ h)^{-1}(P^*)=P$ and $P^*=P_0(Q(A)\otimes Q(B))$.

Remark 2.11: If S is torsion-free as an A and B-module (which holds for instance when D is a field), then $K=0$ and $l\circ h\colon S\to Q(A)\otimes Q(B)$ is an embedding. Then the above correspondence is such that $P=P_0(Q(A)\otimes Q(B))\cap(A\otimes B)$. The representation of A-B-disjoint prime ideals of $A\otimes B$ in this form when D is a field was obtained in ([6], Corollary 1.3). However it was not proved there that this gives a one-to-one correspondence.

Corollary 2.10 gives in particular the following:

COROLLARY 2.12: *The following conditions are equivalent*

(i) *$C(A)\otimes C(B)$ is a domain*

(ii) *The torsion ideal K is a prime ideal of $A\otimes B$.*

In particular, $A\otimes B$ is prime if and only if $A\otimes B$ is torsion-free as an A and B-module and $C(A)\otimes C(B)$ is a domain.

The results on centred extensions give several applications to A-B-disjoint prime ideals of S.

COROLLARY 2.13: *Let P be an A-B-disjoint prime ideal of $A\otimes B$. Then P is strongly prime (resp. non-singular, l.n.-semisimple) if and only if A and B are strongly prime (resp. non-singular, l.n.-semisimple).*

Proof: Let P_0 be the prime ideal of $C(A) \otimes C(B)$ corresponding to P. Since $C(A) \otimes C(B)$ is commutative, P_0 is strongly prime, non-singular as well as l.n.-semisimple. Now the corollary follows from Propositions 1.12 and 1.15, Remark 1.16 and Corollary 1.8. ∎

Similarly, Corollary 1.11 gives

COROLLARY 2.14: *Let P be an A-B-disjoint prime ideal of $A \otimes B$. If A and B are nil-semisimple, then P is nil-semisimple.*

The converse of Corollary 2.14 is not true in general. We have

Example 2.15: Let A be a finitely generated not nilpotent algebra over the field of rational numbers $\mathbf{Q}$ such that $A \otimes_{\mathbf{Q}} K$ is a nil algebra for every field extension $\mathbf{Q} \subseteq K$ (cf. [1], Lemma 59). Since A is finitely generated, there exists an ideal I of A maximal with respect to the property $A^n \not\subseteq I$ for every natural number n. Clearly $B = A/I$ is a prime algebra such that for every $0 \neq J \lhd B$, B/J is nilpotent. Let R be the prime $\mathbf{Q}$-algebra obtained from B by adjoining an identity and $S = R \otimes_{\mathbf{Q}} R$. By [10] the algebra $B \otimes_{\mathbf{Q}} B$ is not nil. Hence there exists a prime nil-semisimple ideal P of S such that $B \otimes_{\mathbf{Q}} B \not\subseteq P$. Note that P is R-R-disjoint. Indeed, if $J = \{r \in R \mid r \otimes 1 \in P\} \neq 0$, then $B^n \subseteq J$ for some n. Thus $B^n \otimes B \subseteq P$ and, since the ideal P is prime, $B \otimes_{\mathbf{Q}} B \subseteq P$, a contradiction. Now the prime ideal P_0 of $C(R) \otimes_{\mathbf{Q}} R$ corresponding to P satisfies $C(R) \otimes B \not\subseteq P_0$ (otherwise we would have $B \otimes B \subseteq P$). Hence both P_0 and R are not nil-semisimple. This provides also a counterexample to the converse of Corollary 1.11.

COROLLARY 2.16: *Let P be an A-B-disjoint prime ideal of $A \otimes B$ and let P_0 be the corresponding prime ideal of $C(A) \otimes C(B)$. Then P_0 is a maximal ideal of $C(A) \otimes C(B)$ if and only if P is maximal among the ideals of $A \otimes B$ not containing non-zero elements of the type $a \otimes b$, $a \in A$, $b \in B$.*

Proof: By Lemma 2.3 P does not contain non-zero elements of the type $a \otimes b$, $a \in A$, $b \in B$. Assume P_0 is maximal and take an ideal M of $A \otimes B$ which is maximal among the ideals containing P and not containing non-zero elements of the type $a \otimes b$, $a \in A$, $b \in B$. By Lemma 2.2, M is a prime A-B-disjoint ideal of $A \otimes B$. Hence M corresponds to a prime ideal M_0 of $C(A) \otimes C(B)$ such that $P_0 \subseteq M_0$. Hence $P_0 = M_0$ and so $P = M$.

Conversely, let M_0 be a maximal ideal of $C(A) \otimes C(B)$ containing P_0 and let M be the corresponding A-B-disjoint prime ideal of $A \otimes B$. By Lemma 2.3 and the maximality of P we have $M = P$ and consequently $M_0 = P_0$. ∎

Corollary 2.16 immediately gives the following generalization of Corollary 1.4 in [6] and the main result of [8].

COROLLARY 2.17: *The following conditions are equivalent:*

(i) *$C(A) \otimes C(B)$ is a field.*

(ii) *The torsion ideal K is maximal among the ideals of $A \otimes B$ not containing non-zero elements of the type $a \otimes b$, $a \in A$, $b \in B$.*

We also have

COROLLARY 2.18: *Assume that A and B are primitive algebras and P is an A-B-disjoint ideal of S which is maximal among the ideals of S not containing non-zero elements of the type $a \otimes b$, $a \in A$, $b \in B$. Then P is a primitive ideal of S.*

Proof: By Lemma 2.2 P is prime. Denote by P_0 the prime ideal of $C(A) \otimes C(B)$ corresponding to P. By Corollary 2.16, the ideal P_0 is maximal. Hence by Proposition 1.21 the ideal P is primitive. ∎

For every commutative ring R, $\beta(R) = L(R) = \mathrm{Nil}(R) = s(R) = z(R)$. Moreover, it is well known that if K_1, K_2 are fields, then $G(K_1 \otimes K_2) = J(K_1 \otimes K_2) = \beta(K_1 \otimes K_2)$. Applying these and results of Section 1 one easily gets

COROLLARY 2.19: *Assume that A and B are prime (resp. prime l.n.-semisimple, prime nil-semisimple, strongly prime, prime non-singular, primitive, simple) rings. Then*

$$\beta(A \otimes B) = (l \circ h)^{-1}((Q(A) \otimes Q(B))\beta(C(A) \otimes C(B)))$$

(resp. $\alpha(A \otimes B) = (l \circ h)^{-1}((Q(A) \otimes Q(B))\beta(C(A) \otimes C(B)))$, for $\alpha = L, \mathrm{Nil}, s, z, J, G$).

3. Tensor products: general case

In this section we show how the results on prime ideals of tensor products of algebras over a commutative ring can be obtained, after some reduction, as an application of the results of the former sections. Throughout A and B are algebras over a commutative ring F and $S = A\otimes_F B$.

Given an ideal I of S, define the ideals I_A and I_B of A and B, respectively, as in Section 2. If P is a prime ideal of S, then P_A and P_B are prime ideals of A and B, respectively. Now we ask for conditions under which for given prime ideals P' and P'' of A and B, respectively, there exists a prime ideal P of S such that $P_A = P'$ and $P_B = P''$.

Denote by $\alpha: F \to A$ (resp. $\beta: F \to B$) the canonical mapping $f \to f1_A$ (resp. $f \to f1_B$). We have

PROPOSITION 3.1: *Let P' and P'' be ideals of A and B, respectively. Then there exists a prime ideal P of S such that $P_A = P'$ and $P_B = P''$ if and only if P' and P'' are prime ideals with $\alpha^{-1}(P') = \beta^{-1}(P'')$.*

Proof: If P is a prime ideal with $P_A = P'$ and $P_B = P''$, then P' and P'' are prime ideals. Moreover $f \in \alpha^{-1}(P') = \alpha^{-1}(P_A)$ if and only if $f \otimes 1 \in P$. Since $f \otimes 1 = 1 \otimes f$, this is equivalent to $f \in \beta^{-1}(P_B) = \beta^{-1}(P'')$.

Conversely, assume that P' and P'' are prime ideals with $\alpha^{-1}(P') = \beta^{-1}(P'')$ and denote this ideal by I. Then I is a prime ideal of F and so $D = F/I$ is a domain such that the natural mappings from D to A/P' and B/P'' are embeddings. By Lemma 2.2 there exists an (A/P')-(B/P'')-disjoint prime ideal $\bar{P}$ of $(A/P')\otimes_D(B/P'')$. Also there exists a natural epimorphism $\phi: S \to (A/P')\otimes_F(B/P'') \simeq (A/P')\otimes_D(B/P'')$. Thus $P = \phi^{-1}(\bar{P})$ is a prime ideal of S with $S/P \simeq ((A/P')\otimes_F(B/P''))/\bar{P}$. Moreover $P_A = P'$. Indeed, $a \in P_A$ if and only if $a \otimes 1 \in P$, i.e., $\bar{a} \otimes \bar{1} \in \bar{P}$, where $\bar{a} = a + P'$ and $\bar{1} = 1 + P''$. Since $\bar{P}$ is (A/P')-disjoint, the last condition is equivalent to $\bar{a} = 0$, i.e., $a \in P'$. Hence $P_A = P'$. Similarly one obtains that $P_B = P''$. ∎

Now we shall obtain a characterization for $A \otimes B$ to be prime.

THEOREM 3.2: *The following conditions are equivalent*

(i) *S is a prime ring.*

(ii) *The ideals 0_A and 0_B are prime, S is torsion-free as $A/0_A$ and $B/0_B$-module and $C(A/0_A)\otimes_F C(B/0_B)$ is a domain.*

Proof: Note that $S \simeq (A/0_A)\otimes_F(B/0_B) \simeq (A/0_A)\otimes_D(B/0_B)$, where $D = F/\alpha^{-1}(0_A)$. Now the theorem is a direct consequence of Corollary 2.12. ∎

Remark 3.3: If A and B are prime F-algebras and $\operatorname{Ker}\alpha = \operatorname{Ker}\beta$, then $D = F/\operatorname{Ker}\alpha$ embedds into A and B and $A\otimes_F B \simeq A\otimes_D B$. In this case $0_A = 0_B = 0$ and the results of Lemmas 2.1 and 2.2 hold.

Now we assume that P is a prime ideal of S and we show how some results on the factor S/P can be obtained by reducing to the disjoint case. Denote by I the prime ideal $\alpha^{-1}(P_A) = \beta^{-1}(P_B)$ of F and put $D = F/I$. Note that $(A/P_A)\otimes_F(B/P_B) \simeq (A/P_A)\otimes_D(B/P_B)$ and the canonical mappings $D \to A/P_A$ and $D \to B/P_B$ are embeddings.

Let ϕ: $S \to (A/P_A)\otimes_F(B/P_B)$ be the epimorphism of rings defined by $\phi(a\otimes b) = (a+P_A)\otimes(b+P_B)$. It is clear that $L = A\otimes P_B + P_A\otimes B = \operatorname{Ker}\phi$ and $S/L \simeq (A/P_A)\otimes_F(B/P_B)$. Now $L \subseteq P$ and consequently $\phi(P)$ is a prime ideal of $(A/P_A)\otimes_F(B/P_B)$ such that $S/P \simeq ((A/P_A)\otimes(B/P_B))/\phi(P)$. We easily see that $\phi(P)$ is an (A/P_A)-(B/P_B)-disjoint ideal of $(A/P_A)\otimes(B/P_B)$.

The foregoing observations and Corollaries 2.13 and 2.14 immediately give

THEOREM 3.4: *Let P be a prime ideal of $S = A\otimes_F B$. Then P is strongly prime (resp. non-singular, l.n.-semisimple) if and only if P_A and P_B are strongly prime (resp. non-singular, l.n.-semisimple) ideals of A and B, respectively.*

THEOREM 3.5: *If P is a prime ideal of S such that P_A and P_B are nil-semisimple ideals of A and B, respectively, then P is also nil-semisimple.*

As a consequence of Corollary 2.18 we have

THEOREM 3.6: *Assume that P' and P'' are primitive ideals of A and B, respectively, with $\alpha^{-1}(P') = \beta^{-1}(P'')$ and let P be an ideal of S which is maximal among ideals I of S with $I_A = P'$ and $I_B = P''$. Then P is a primitive ideal of S.*

Proof: Using the above reduction we may assume $P' = P'' = 0$. Take an ideal I of S which is maximal with respect to the following condition: $P \subseteq I$ and I contains no non-zero elements of the type $a\otimes b$, where $a \in A$ and $b \in B$. Then I is primitive by Corollary 2.18 and by the maximality of P we have $I = P$. ∎

Applying Corollary 1.23 one obtains

COROLLARY 3.7: *If A and B are F-algebras such that B is free as an F-module, $\alpha(A) = 0$ and $\alpha(C\otimes_F B) = 0$ for every field C which is the extended centroid of a prime factor of A, then $\alpha(S) = 0$ provided α is any of the radicals $s, z, \beta, L, \mathrm{Nil}, J, G$.*

In particular we have (see [7])

COROLLARY 3.8: *Assume that A and B are algebras over a field F and α is any of the radicals in Corollary 3.7. If $\alpha(A) = \alpha(B) = 0$ and $\beta(C(\bar{A})\otimes_F C(\bar{B})) = 0$ for all prime factors $\bar{A}$ and $\bar{B}$ of A and B, respectively, then $\alpha(A\otimes_F B) = 0$.*

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